UDC 531.36

QUALITATIVE INVESTIGATION OF A PARTICULAR NONLINEAR SYSTEM

PMM Vol. 39, № 4, 1975, pp. 633-641

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(Received January 10, 1975)

Methods of bifurcation theory and characteristics approximated by cubic polynomials of a general form are used for qualitative investigation of a system which has interesting applications. Possible bifurcations and the behavior of bifurcation curves are considered.

The considered system occurs in investigations of processes in electrical circuits containing negative nonlinear resistors [1], tunnel diode circuits [2−4] and in problems of compressor surge with a linear approximation of throttling [5]. A complete qualitative investigation of such system with piecewise-linear approximation is given in [6].

1. The equation of motion. We consider the system

$$x^{\cdot} = y - \varphi(x) \equiv P, \quad y^{\cdot} = \sigma - \lambda x - y \equiv Q \quad (\sigma > 0,\ \lambda > 0) \tag{1.1}$$

where $\varphi(x)$ has a downward sloping section and is approximated by the cubic polynomial $\varphi(x) = ax^3 - bx^2 + cx$ with conditions

$$a > 0, \quad b > 0, \quad c > 0, \quad b^2 - 3ac > 3a \tag{1.2}$$

The latter is equivalent to the condition $\min \varphi'(x) < -1$, for which various bifurcations are possible in the system. For $\min \varphi'(x) > -1$ the appearance and disappearance of the equilibrium state are the only possible bifuractions, since over the whole plane $P_x' + Q_y' \neq 0$.

2. Equilibrium states and their bifurcation. One or three equilibrium states are possible. In the case of a single equilibrium state we have a stable focal point (node), if at the point of intersection of isoclines we have $\varphi'(x) > -1$, while in the opposite case the focal point is unstable. In the case of three equilibrium states we have a saddle point between focal points (nodes).

The discriminant curve which separates in the $\lambda\sigma$-plane the region of three equilibrium states from that of single equilibrium is determined by the condition of tangency of the straight line $y = \sigma - \lambda x$ to the curve $y = \varphi(x)$. It is defined by the parametric equations

$$\sigma = \varphi(x_0) - x_0\varphi'(x_0) = -2ax_0^3 + bx_0^2 \tag{2.1}$$
$$\lambda = -\varphi'(x_0) = -3ax_0^2 + 2bx_0 - c$$

where x_0 is the coordinate of the tangency point.

From the condition $\varphi''(x_0) = 0$ we have $x_0 = b / 3a$ which defines a cuspidal point of the discriminant curve. Its coordinates are

$$\lambda_2 = (1/_3\, a)(b^2 - 3ac), \qquad \sigma_2 = b^3 / 27a^2 \tag{2.2}$$

The discriminant curve lies to the left of the cuspidal point and is convex relative to the equilibrium states. Two equilibrium states of system (1.1), viz a focal point (node) and a complex saddle-node equilibrium state, correspond to the points of the discriminant curve, while the cuspidal point is associated with the merging of the three equilibrium states.

Elimination of parameter x_0 from (2.1) yields for the discriminant curve an equation of the form

$$\Delta \equiv 27a^2\sigma^2 - 18ab\,(c+\lambda)\sigma + 4b^3\sigma + 4a\,(c+\lambda)^3 - b^2(c+\lambda)^2 = 0$$

For $\lambda = 1$ system (1.1) has a complex focal point on straight lines tangent to the discriminant curve and directed into the region of $\lambda > 1$.

For the coordinates of equilibrium states we have the equation

$$ax^3 - bx^2 + (c+\lambda)x - \sigma = 0 \tag{2.3}$$

The condition $P_x' + Q_y' = 0$ yields $3ax^2 - 2bx + c + 1 = 0$, hence

$$x_{1,2} = (1/3a)\,[b \pm \sqrt{b^2 - 3a\,(c+1)}], \quad x_1 < x_2 \tag{2.4}$$

Substituting (2.4) into (2.3) for the equilibrium state at $x = x_1$ we obtain

$$L_1 \equiv 9ab\,(c+\lambda) - 2b^3 - 27a^2\sigma - (b^2 - 3ac - 3a)^{1/2}\,(6ac - 2b^2 - 3a + 9a\lambda) = 0 \tag{2.5}$$

and for the equilibrium state at $x = x_2$

$$L_2 \equiv 9ab\,(c+\lambda) - 2b^3 - 27a^2\sigma + (b^2 - 3ac - 3a)^{1/2} \times (6ac - 2b^2 - 3a + 9a\lambda) = 0 \tag{2.6}$$

The straight lines (2.5) and (2.6) are tangent in the parameter plane $\lambda\sigma$ to the upper and lower branches of the discriminant curve and for $\lambda = 1$ intersect at point $\lambda_0 = (2b^2 + 3a - 6ac)\,/\,9a$, $\sigma_0 = b\,(c+1)\,/\,9a$, while for $\lambda = \lambda_1$ they intersect branches of the discriminant curve

$$\lambda_1 = (b^2 - 3ac + a)\,/\,4a \tag{2.7}$$

Since coordinates (x_1) and (x_2) may define a focal point $(\lambda > 1)$ or a saddle point $(\lambda < 1)$, hence the focal or saddle parameter $P_x' + Q_y'$ may change sign at the crossing of straight lines (2.5) and (2.6).

For a complex focal point the first Liapunov parameter for system (1.1) is of the form [8]

$$\alpha_3 = (\pi\,/\,4)(\lambda - 1)^{-3/2}\;[3a\,(\lambda + 1) - 2\,(b^2 - 3ac)] \tag{2.8}$$

For $\lambda = \lambda_3, \lambda_3 = (2\,/\,3a)\,(b^2 - 3ac) - 1$ parameter α_3 vanishes.

A complex focal point is stable for $\lambda > \lambda_3$ $(\alpha_3 < 0)$ and unstable for $\lambda < \lambda_3 (\alpha_3 > 0)$. For $\lambda = \lambda_3$ $(\alpha_3 = 0)$ the stability of a complex focal point is determined by the sign of the second Liapunov parameter α_5 which can be calculated by the readily available formula [7]

$$\alpha_5 = -\,{}^5/_8\;\pi a^2\,(\lambda - 1)^{-3/2} < 0 \tag{2.9}$$

Note that for α_3 and α_5 the coordinate of equilibrium state does not appear in formulas (2.8) and (2.9) hence for parameter values on the straight lines (2.5) and (2.6) for $\lambda > 1$, the derived formulas apply equally to the left- (x_1) and the right-hand (x_2) complex focal points.

3. Behavior at infinity. Let us plot in the xy-plane a rectangle with sides parallel to coordinate axes and whose diagonal is the isocline $y = \sigma - \lambda x$. For a fairly large rectangle the isocline $y = \varphi(x)$, whose order of increase is higher than that of a straight line, does not intersect its sides parallel to the y-axis, but intersects each of the other two sides once. With increasing t all trajectories of system (1.1) enter such rectangle. For any values of the system parameters infinity is unstable (Fig. 1).

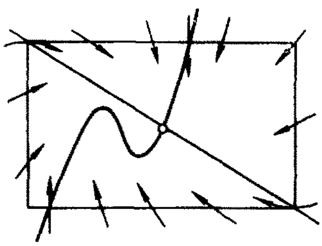

Fig. 1

4. Qualitative determination of the pattern of phase space subdivision.

4.1. Symmetry in the phase space. By transferring the coordinate origin of the point of inflection of the characteristic $y = \varphi(x)$ we reduce system (1.1) to the form

$$\xi^{\cdot} = A\xi + \eta - a\xi^3, \quad \eta^{\cdot} = \bar{\sigma} - \lambda\xi - \eta \tag{4.1}$$
$$A = (b^2 - 3ac) \quad 3a, \quad \bar{\sigma} = \sigma - (1/27a^2)[9ab(c+\lambda) - 2b^3]$$

Formula (4.1) implies that (a) if $\bar{\sigma} = 0$, the phase space of system (4.1) is symmetric with respect to the coordinate origin (the point of inflection of the characteristic), and (b) if straight lines $\eta = \bar{\sigma}_1 - \lambda\xi$ and $\eta = \bar{\sigma}_2 - \lambda\xi$ are symmetric with respect to the coordinate origin ($\bar{\sigma}_1 + \bar{\sigma}_2 = 0$), the phase patterns for $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are symmetric to each other about the inflection point of the characteristic.

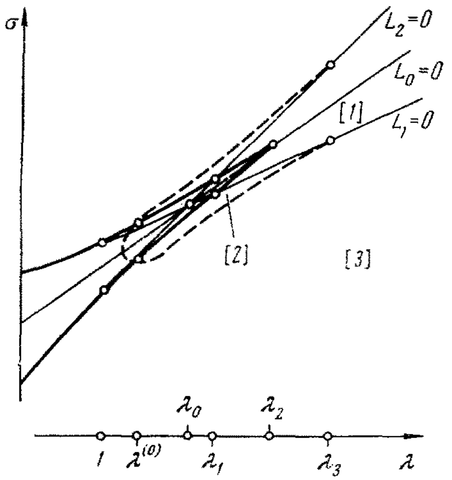

Fig. 2

Owing to this it is possible to confine the analysis of the parameter space $\lambda\sigma$ to its part lying either above or below the line of symmetric patterns $L_0 \equiv \bar{\sigma} = 0$.

The discriminant curve, the straight lines of stability change $L_1 = 0$ and $L_2 = 0$ (for focal points x_1 and x_2, respectively), and the line $L_0 = 0$ which passes through the intersection point of L_1 and L_2 are shown in Fig. 2.

The subsequent analysis is carried out for values of parameters below the line of symmetric patterns.

4.2. The pattern of phase space subdivision and bifurcation related to parameter variation along the straight line of stability change of focal point x_1. Let us examine the bifurcations and qualitative variation of the pattern of phase space subdivision with variation of parameters along the straight line $L_1 = 0$.

If $\lambda > \lambda_3$, then $\alpha_3 < 0$ and the complex focal point x_1 is stable. Assuming that $b^2 - 3ac - 3a > 0$ (see (2.4)), we obtain from the formula for λ_3 and (2.7) that $\lambda_3 > \lambda_1$.

Hence for these values of parameters system (1.1) has a single equilibrium state. If λ decreases beyond $\lambda = \lambda_3$, the sign of α_3 and the stability of the complex focal point change (although the latter remains complex), and a stable limit cycle is generated at the focal point. This stable cycle remains unaffected by further decrease of λ within the interval $\lambda_1 < \lambda < \lambda_3$. The value $\lambda = \lambda_1$ corresponds to the tangency of straight line $\sigma - \lambda x - y = 0$ to the characteristic $y = \varphi(x)$. A saddle-node with an unstable nodal region (in the saddle-node $(P_x' + Q_y' = -\varphi'(x_0) - 1 > 0$ appears in the phase plane. It can be verified that this occurs precisely inside the limit cycle for any characteristics which correspond to the particular selection of coefficients of $\varphi(x)$.

If for some approximations the saddle-node appears inside the cycle and for others outside the latter, then, because of continuity, a characteristic must exist for which the saddle-node occurs in the limit cycle. A saddle-node with an unstable nodal region cannot, however, occur in a stable limit cycle [8].

Thus for any one specific approximation it is sufficient to know the relative disposition of the cycle and saddle-node. It was established by numerical methods that for system (1.1) with approximation $\varphi(x) = \frac{1}{3}x^3 - 3x^2 + 7x$ and parameters $\sigma = \frac{49}{6}$ and $\lambda = \frac{7}{4}$ (which corresponds to the equilibrium state of a complex focal point and a saddle-node) the cycle contains a saddle-node. Hence this applies to any cubic approximation.

Further movement along the straight line L_1 inside the region bounded by the discriminant curve results in the splitting of the saddle-node into a saddle and an unstable node which is subsequently transformed into a focal point. Within the interval $\lambda_0 < \lambda < \lambda_1$ bifurcations of the equilibrium state are absent, and bifurcation of separatrices is inhibited by the sign of the saddle parameter. (Loops of separatrices, if they occur, must be stable, but this is impossible, since at the saddle point the parameter $P_x' + Q_y'$ is positive [8].) Consequently a phase space containing two complex focal points symmetric with respect to the saddle corresponds to point $\lambda = \lambda_0$ of intersection of the straight line $L_1 = 0$ with the line of symmetric patterns (the α-separatrices lead to a stable cycle which comprises all three stable states and the ω-separatrices are formed by twists of unstable complex focal points.

Note. Qualitatively the pattern of phase space subdivision into trajectories, determined on the basis of the above information, is accurate only within the supplementary even number of limit cycles, which may possibly be due to bunching of trajectories. This incompleteness cannot be eliminated in the subsequent analysis.

4.3. The pattern of phase space subdivision and bifurcation related to parameter variation along the line of symmetric patterns. Let us consider bifurcations and phase space changes along the line of symmetric patterns $L_0 = 0$. If $\lambda > \lambda_2$ (λ_2 is defined by formula (2.2)), an unstable focal point (node) represents the only equilibrium state of the system, with an unstable infinity. A stable limit cycle exists around the focal point. The decrease of λ along the straight line $L_0 = 0$ results in the rotation of line $y = \sigma - \lambda x$ around the equilibrium state at the inflection point of the characteristic $y = \varphi(x)$. For $\lambda = \lambda_2$ the straight line is tangent to the characteristic at the inflection point ($L_0 = 0$ intersects the discriminant curve at the cuspidal point). This results in a complex equilibrium state which with further decrease of λ decomposes into three simple states, viz two unstable focal points (nodes) with a saddle point between these. Bifurcations of the equilibrium state do not occur in the interval $\lambda_0 < \lambda < \lambda_2$. For $\lambda = \lambda_0$ both focal points become complex, and with further decrease of λ they generate unstable limit cycles (the first focal parameter α_3 is positive). A phase space pattern with three limit cycles is obtained, the α-separatrices lead to a stable cycle which comprises all three equilibrium states, while the ω-separatrices are formed by twists of unstable cycles which comprise stable focal points (Fig. 3 (9)).

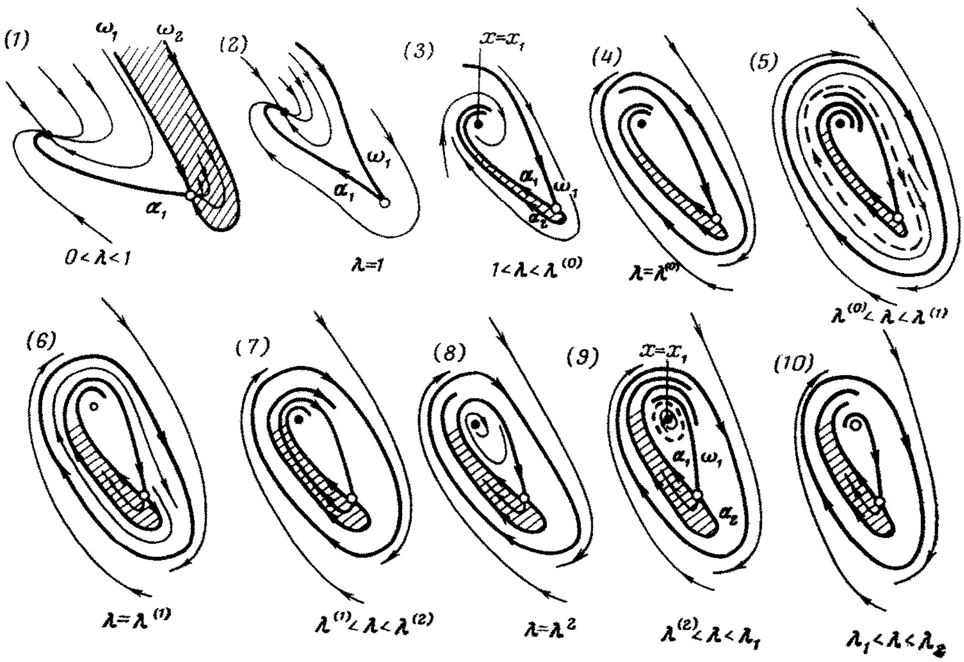

Fig. 3

A further decrease of λ in the interval $0 \leqslant \lambda < \lambda_0$ does not lead to any change of the equilibrium state stability, and for $\lambda = 0$ cycles are absent (there exists then the integral straight $y = \sigma$ which passes through all equilibrium states). Limit cycles can only vanish either by conversion into loops of separatrices or by merging with cycles newly created from such loops. It is significant that cycles around focal points and those which comprise all three equilibrium states have different stabilities. Depending on the sign of the saddle parameter, only unstable cycles comprising equilibrium states may convert (and for some $\lambda = \lambda^+$ are necessarily converted) into separatrix loops [8]. These two loops (which appear simultaneously, because $L_0 = 0$ is the line of symmetric patterns) form a separatrix contour in the shape of an "eight" whose disintegration (with decreasing λ) generates an unstable limit cycle comprising three equilibrium states. For some $\lambda = \lambda^* < \lambda^+$ these limit cycles merge and vanish with decreasing λ.

4.4. The pattern of phase space subdivision and bifurcation related to parameter variation along the discriminant curve. Let us examine the bifurcations and changes of phase space patterns along the lower branch of the discriminant curve beginning at the cuspidal point $\lambda = \lambda_2$. In the interval $\lambda_1 < \lambda < \lambda_2$ we have a pattern containing an unstable focal point and a saddle-node with unstable nodal region inside the stable limit cycle (Fig. 3 (10)). Stability of the focal point x_1 changes with λ decreasing beyond $\lambda = \lambda_1$ and, since $\alpha_3 > 0$, that point generates an unstable limit cycle (Fig. 3 (9)). To examine further bifurcations occurring for λ tending to zero it is necessary, first of all, to determine the pattern for $\lambda = 0$. This is readily obtained, since for $\lambda = 0$ we have the integral straight line $y = \sigma$ which passes through both equilibrium states (stable node and a saddle-node with stable nodal region). There are no limit cycles, and tha pattern is qualitatively equivalent to that shown in Fig. 3 (1) (the nodal region is hatched). For a saddle-node $\lambda = 1$ implies bifurcation. For $\lambda < 1$ the nodal region is stable (the saddle-node has two α-separatrices), for $\lambda > 1$ it is unstable (the saddle-node has two ω-separatrices), and for $\lambda = 1$ the saddle-node degenerates (the characteristic equation has two zero roots) and the nodal region vanishes (the saddle-node has one α- and one ω-separatrix). Within the interval $0 < \lambda < 1$ the pattern of equilibrium states is retained and is similar to that shown in Fig. 3 (1).

A qualitative determination of the pattern for $\lambda = 1$ is essential for the examination of the sequence of bifurcations along the discriminant curve. As shown below, the increase of λ beyond zero must unavoidably result in a two-limit cycle because of trajectory bunching. Such cycle comprises equilibrium states, but there are no means for the exact determination of parameters for which it occurs. Below we assume that limit cycles do not yet appear for $\lambda = 1$ and that the pattern shown in Fig. 3 (2) applies (results related to the assumption of existence of limit cycles for $\lambda = 1$ are presented subsequently).

For λ increasing beyond $\lambda = 1$ the pattern is qualitatively equivalent to that shown in Fig. 3 (3). An unstable nodal region of the saddle-node is generated (both α-separatrices emerge in the direction $\varkappa = -1$, and the ω-separatrix enters in the direction $\varkappa = -\lambda$). The node becomes a focal point for $[1 - \varphi'(x_1)]^2 - 4\lambda < 0$, where $\varphi'(x_1) = 1/a\,(b^2 - 3\,ac - 4\,\lambda)$.

Let us now compare the disposition of α- and ω-separatrices in the patterns shown in Figs. 3 (9) and 3 (3). On the segment of the straight line $x = x_1$ lying above the focal point we mark the points of its intersection with the α- and the ω-separatrices (nearest to the saddle-node along the separatrix path). For the pattern shown in Fig. 3 (9) the trace of the ω-separatrix on the line $x = x_1$ lies below the traces of α-separatrices, while for the pattern in Fig. 3 (3) it lies above the latter. The decrease of λ must necessarily lead to successive bifurcations associated with the merging of the ω-separatrix trace on line $x = x_1$ with the trace of the α_1-separatrix (emerging upward from the saddle-node) and of the α_2-separatrix (emerging downward). Since the saddle parameter $P_x' + Q_y' = \lambda - 1$ for $\lambda > 1$ is positive, the formation of the first loop (for $\lambda = \lambda^{(2)}$) contracts to it unstable limit cycles [8] (Fig. 3 (8)). When the trace of the ω-separatrix lies between the traces of the α_1- and α_2-separatrices, a closed contour is formed by the separatrix of the saddle-node (Fig. 3 (7)). A separatrix loop develops when traces of the ω- and α_2-separatrices coincide (for $\lambda = \lambda^{(1)} < \lambda^{(2)}$) (Fig. 3 (6)). Disintegration of this loop with decreasing λ generates an unstable limit cycle which comprises both equilibrium states, yielding the pattern shown in Fig. 3 (5), which has two limit cycles between which there are no equilibrium states. Within the interval $1 < \lambda < \lambda^{(1)}$ at certain $\lambda = \lambda^{(0)}$ limit cycles merge into a double semistable limit cycle (Fig. 3(4) and then vanish. The sequence of patterns along the lower branch of the discriminant curve is shown in Fig. 3.

4.5. The pattern of phase space subdivision and bifurcation in the region of three equilibrium states inside of the discriminant curve. The discriminant curve defined by parametric equations (2.1) can be considered as the envelope of the set of straight lines $\sigma - \lambda x_0 - \varphi(x_0) = 0$ lying in the $\lambda\sigma$-plane. The half-lines

$$\sigma - \lambda x_0 - \varphi(x_0) = 0,\ \lambda \leqslant -\varphi'(x_0) \qquad (4.2)$$

tangent to the discriminant curve at point $\lambda = -\varphi'(x_0)$ pass once through the region bounded by the line of symmetric patterns and the lower branch of the discriminant curve, when x_0 is varied between the inflection point and the minimum of the characteristic $\varphi(x)$ (for $b/3a \leqslant x_0 \leqslant (1/3a)(b+\sqrt{b^2-3ac})$). In the parameter plane $\lambda\sigma$ the motion along half-lines (4.2) from the tangency point corresponds for system (1.1) to a counterclockwise rotation of the isocline $\sigma - \lambda x - y = 0$ about the saddle point which appears at point $x = x_0$ of splitting of the saddle-node into saddle and node.

Let us consider the bifurcations generated by the motion along half-lines (4.2) tangent to the discriminant curve in the interval $\lambda_1 < \lambda < \lambda_2$, where bifurcations of equilibrium states, of separatrices, and of limit cycles are present.

With decreasing λ the saddle-node equilibrium state within a stable limit cycle splits into a saddle and an unstable node which with further decrease of λ is transformed into a focal point (Fig. 5 (11)). At intersection points of the considered half-line (4.2) with the straight lines $L_1 = 0$ and $L_2 = 0$ the equilibrium states bifurcate as follows: with decreasing λ unstable limit cycles are generated, first, at the focal point x_1 (Fig. 5 (10)), then from the focal point x_2 (focal points become stable), which yields a pattern with three limit cycles. Since for $\lambda = 0$ limit cycles are absent ($y = \sigma$ is the integral straight line and the pattern is equivalent to that shown in Fig. 5 (1)). Hence, using a reasoning similar to that in Sect. 4.4, we find that for λ tending to zero the following bifurcations of separatrices must exist: a separatrix loop around the upper focal point and a "large loop" around the lower focal point with two equilibrium states within the latter. Since the saddle parameter $(P_x' + Q_y' = -\varphi'(x_0) - 1 = \lambda - 1$, where λ is the coordinate of the tangency point of half-line (4.2) with the discriminant curve, separatrix loops can only be unstable, and their generation is accompanied by merging with (or conversely, emanation from) unstable limit cycles. Separatrix loops appear around focal points as the result of merging with them of unstable limit cycles emanating from focal points. Disintegration of a "large loop" formed by α_2- and ω_1-separatrices of the saddle point are accompanied by the appearance of an unstable limit cycle which comprises all equilibrium states (the large loop cannot be the result of contraction to it of a stable limit cycle, since

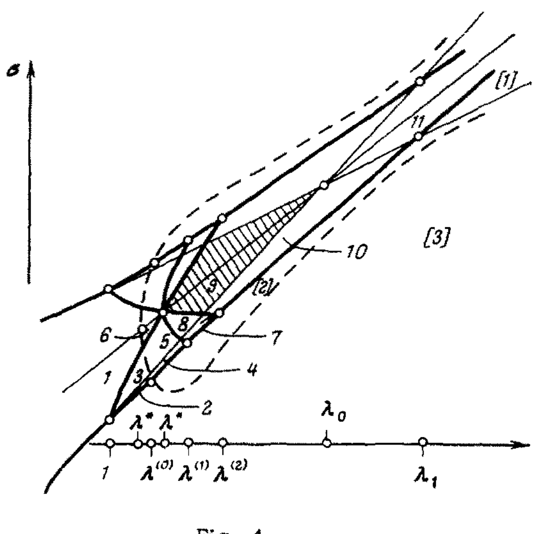

Fig. 4

this is inhibited by the sign of the saddle parameter [8]). For $\lambda = 0$ cycles are absent, hence with further decrease of λ the merging of a stable cycle with an unstable one must result in the emergence of a binary limit cycle followed by its disappearance. Bifurcation points on half-lines (4.2) associated with separatrix loops around the upper and lower focal points may either coincide or be separated by the bifurcation point corresponding to the large loop.

For half-lines (4.2) tangent to the lower boundary of the discriminant curve within the intervals $(\lambda^{(2)}, \lambda_1)$, $(\lambda^{(1)}, \lambda^{(2)})$, $(\lambda^{(0)}, \lambda^{(1)})$ and $(1, \lambda^{(0)})$ bifurcations are similar, however their number decreases from interval to interval, because some of the bifurcations had already appeared along the path of the discriminant curve.

Since the above bifurcations occur at straight lines which completely fill the region inside of the discriminant curve, there must exist continuous curves along which bifurcation takes place. Their initial and end points lie on the line of symmetric patterns and on the discriminant curve. All three bifurcation curves associated with the three kinds of separatrix loops intersect at point $\lambda = \lambda^+$ on the line of symmetric patterns (Fig. 4), and end at points $\lambda = \lambda^{(2)}$, $\lambda = \lambda^{(1)}$ and $\lambda = 1$ of the discriminant curve.

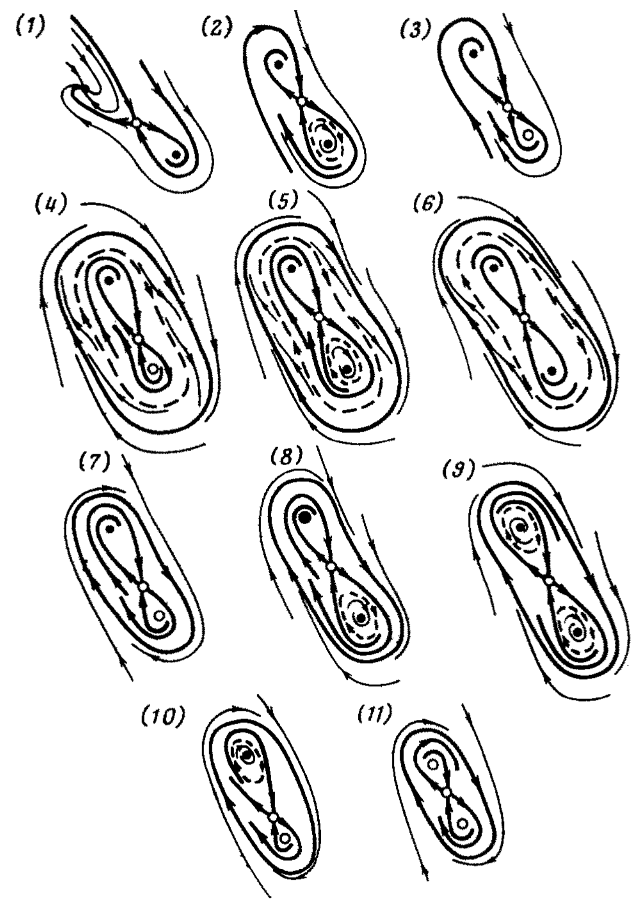

Fig. 5

Patterns with separatrix loops of the degenerated saddle-node must be considered as the degeneration of a separatrix loop lying around the upper focal point and contracted into the latter. The binary cycle curve passes through point $\lambda = \lambda^*$ of the line of symmetric patterns and through point $\lambda = \lambda^{(0)}$ of the discriminant curve to the left of the curve with the large loop.

Since some of the bifurcation curves may intersect each other, the sequence of patterns and bifurcations can be qualitatively different for the various λ-parameters along segments of tangents lying inside the discriminant curve.

4.6. The pattern of phase space subdivision and bifurcation outside the discriminant curve (in the region of single equilibrium space). Three patterns are possible here: an unstable focal point inside a stable limit cycle, a stable focal point surrounded by two limit cycles, and a stable focal point (node) to which trajectories converge from infinity. The first of these exists at points outside the discriminant curve in the region between the straight line $L_1=0$ and the line of symmetric patterns (region [1] in Figs. 2 and 4). In the interval $\lambda^{(0)} < \lambda < \lambda_3$ the region of existence of two limit cycles adjoins a piece of the discriminant curve and the straight line $L_1 = 0$. A displacement in the interval $\lambda^{(0)} < \lambda < \lambda_1$ from the discriminant curve (Figs. 3 (5) – 3 (9)) into the region of single equilibrium state results in the disappearance of the saddle-node equilibrium state, leaving two limit cycles around a stable focal point (the unstable limit cycle in Figs. 3 (6) – 3 (8) develops from the closed trajectory of the ω-separatrix of the saddle node formed at the disappearance of the latter). In the interval $\lambda_1 < \lambda < \lambda_3$ at transition from region $L_1 < 0$ to region $L_1 > 0$ (with decreasing σ) a second unstable limit cycle ($\alpha_3 > 0$ for $\lambda < \lambda_3$) emerges from the focal point. The bifurcation curve associated with the merging of stable and unstable limit cycles begins at point $\lambda = \lambda_3$ of straight line $L_1 = 0$ (where $\alpha_3 = 0$) and intersects the discriminant curve at $\lambda = \lambda^{(0)}$, thus separating a certain neighborhood of the discriminant curve and of the straight line $L_1 = 0$ at whose points there exists one stable equilibrium state and two limit cycles (region [2] in Figs. 2 and 4). At transition from region $L_1 < 0$ to region $L_1 > 0$ the stable cycle contracts for $\lambda > \lambda_3$ to the focal point ($\alpha_3 < 0$), and we obtain a pattern without limit cycles (region [3] in Figs. 2 and 4). The boundary of the region free of limit cycles consists of a piece of the discriminant curve (for $0 \leqslant \lambda \leqslant \lambda^{(0)}$), the curve of binary cycles (for $\lambda^{(0)} < \lambda < \lambda_3$), and the straight line $L_1 = 0$ (for $\lambda > \lambda_3$).

5. Subdivision of the parameter space. The subdivision of parameter space into regions of qualitatively different patterns on both sides of the line of symmetric patterns $L_0 = 0$ is shown in Fig. 4. The rough patterns associated with the various regions of phase space subdivisions are shown in Fig. 5 (denoted by previously used numerals). Separatrices and limit cycles are shown by heavy lines, unstable limit cycles by dash lines, and stable and unstable equilibrium states are denoted by black dots and small circles, respectively. The region of existence of three limit cycles is shaded in Fig. 4.

The subdivision shown in Fig. 4 is based on the assumption made in Sub-Sect. 4.4 about the absence of limit cycles in the pattern at point $\lambda = 1$ of the discriminant curve. If that assumption is rejected, the binary cycle curve (the dash line in Fig. 4) intersects the discriminant curve at point $\lambda = \lambda^{(0)} < 1$, and the regions [2] and [3] associated with patterns (2) and (3) in Fig. 5 vanish in Fig. 4.

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Translated by J. J. D.

UDC 531.36

QUALITATIVE INVESTIGATION OF A SYSTEM OF THREE DIFFERENTIAL EQUATIONS IN THE THEORY OF PHASE SYNCHRONIZATION

PMM Vol. 39, № 4, 1975, pp. 642-649

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(Received July 1, 1974)

A nonlinear system of three differential equations with three parameters defining the dynamics of a search system for phase synchronization is considered. Qualitative analysis of the system is carried out with the use of Liapunov functions, systems of matching, surfaces without contact, and local theory of bifurcation of multidimensional dynamic systems. It is established that the effective parameter range is determined by the bifurcation of the saddle separatrix loop.

1. Introduction. We consider a system of three differential equations of the form

$$\varphi^{\cdot} = y, \quad y^{\cdot} = v - \frac{a}{b}[F(\varphi) - \gamma], \quad v^{\cdot} = -\frac{1}{b}v + \frac{1}{b}\left(\frac{a}{b} - 1\right)[F(\varphi) - \gamma] \qquad (1.1)$$

where a, b and γ are parameters of function $F(\varphi) \in c^k$ $(k \geqslant 2)$. The system satis-